For online publication: Appendix to "State-dependent local projections"*

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This online appendix consists of four appendices. Appendix A contains the proofs of the main propositions in the paper. Appendix B provides additional theoretical results for a multivariate state-dependent structural VAR model when H_t is exogenous. These results generalize Propositions 3.1 and 3.2 in the main text to a multivariate setting where ε_{1t} is identified within the structural VAR model. Appendix C describes the parameter values used in the data generating process of Section 5. Finally, Appendix D contains additional simulation results.

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A Proofs of the main propositions

The proof of our results relies on the independence between the potential outcomes $y_{t+h}(e)$ and the structural error ε_{1t}. This independence condition follows straightforwardly from our assumptions and is instrumental in providing a causal interpretation to the state-dependent LP estimands. We summarize this result in the following lemma.

Lemma A.1 *Consider the structural process defined by equations (3) and (4) in the main text. Under Assumptions 1 and 2, ε_{1t} is independent of $\{y_{t+h}(e),\ e \in A\}$, where A is the support of ε_{1t}.*

Proof of Lemma A.1. This proof is obvious given the definitions of $y_{t+h}(e)$ derived in the main text. ■

Proof of Lemma 2.1. Let $y_{t+h}(e) = m_h(e, U_{t+h})$. For given e, we can write

$$E\left(y_{t+h}(e+\delta) - y_{t+h}(e)\,|H_{t-1} = \bar{h}\right) = \int \left[m_h(e+\delta, U) - m_h(e, U)\right] f\left(U|\bar{h}\right) dU,$$

where $f\left(U|\bar{h}\right)$ denotes the conditional density function of U_{t+h} given $H_{t-1} = \bar{h}$. Dividing by δ and integrating with respect to e yields

$$\int_A \delta^{-1} E\left(y_{t+h}(e+\delta) - y_{t+h}(e)\,|H_{t-1} = \bar{h}\right) f\left(e|\bar{h}\right) de = \int_A \int \delta^{-1}\left[m_h(e+\delta, U) - m_h(e, U)\right] f\left(U|\bar{h}\right) f\left(e|\bar{h}\right) dU,$$

where $f\left(e|\bar{h}\right)$ denotes the conditional density function of ε_{1t} given $H_{t-1} = \bar{h}$. Under the assumption that ε_{1t} and U_{t+h} are independent, conditionally on $H_{t-1} = \bar{h}$, we have that $f\left(e, U|\bar{h}\right) = f\left(U|\bar{h}\right) f\left(e|\bar{h}\right)$. Moreover, for fixed e and U, by the definition of a derivative, $\lim_{\delta\to 0} \delta^{-1}\left[m_h(e+\delta, U) - m_h(e, U)\right] = m_h'(e, U)$, assuming the derivative of m_h with respect to e exists. Thus,

$$\begin{aligned}
& \lim_{\delta\to 0} \delta^{-1} CAR_h\left(\delta, \bar{h}\right) \\
= & \lim_{\delta\to 0} \int_A \delta^{-1} E\left(y_{t+h}(e+\delta) - y_{t+h}(e)\,|H_{t-1} = \bar{h}\right) f\left(e|H_{t-1} = \bar{h}\right) de \\
= & \int_A \int_U m_h'(e, U) f\left(e, U|\bar{h}\right) de dU \\
= & E\left(m_h'(\varepsilon_{1t}, U_{t+h})\,|H_{t-1} = \bar{h}\right) = E\left(y_{t+h}'(\varepsilon_{1t})\,|H_{t-1} = \bar{h}\right) \equiv CMR_h\left(\bar{h}\right),
\end{aligned}$$

where the last equality follows by definition of $y_{t+h} = m_h(\varepsilon_{1t}, U_{t+h})$. ■

Proof of Proposition 3.1. The proof is in the text. ■

Proof of Proposition 3.2. The proof is in the text. ■

Proof of Proposition 3.3. We start by deriving the potential outcomes $y_{t+h}(e)$ for this model.

For any e, define

$$\beta(e) = \beta_R + (\beta_E - \beta_R)\,\eta(e) \text{ and } \gamma(e) = \gamma_R + (\gamma_E - \gamma_R)\,\eta(e),$$

with $\eta(e) = 1(e > c)$ for any fixed e. Let $V_{0t} \equiv \gamma_{t-1}y_{t-1} + \varepsilon_{2t}$ be a function of $(\varepsilon_{2t}, y_{t-1}, \varepsilon_{1t-1}) = (\varepsilon_{2t}, z'_{t-1}) \equiv U'_t$, since $x_t = \varepsilon_{1t}$ and $z'_t = (x_t, y_t)$. With this notation, for $h = 0$, $y_t = \beta_{t-1}\varepsilon_{1t} + V_{0t}$. The potential outcome for $h = 0$ is obtained from this equation by fixing $\varepsilon_{1t} = e$:

$$y_t(e) = \beta_{t-1}e + V_{0t} \equiv m_0(e, U_t),$$

with $U_t \equiv (\varepsilon_{2t}, z'_{t-1})'$. For $h = 1$, $y_{t+1} = \beta_t\varepsilon_{1t+1} + \gamma_t y_t + \varepsilon_{2t+1}$, where $y_t = y_t(\varepsilon_{1t})$, $\beta_t = \beta(\varepsilon_{1t})$ and $\gamma_t = \gamma(\varepsilon_{1t})$. Hence, upon fixing $\varepsilon_{1t} = e$, we have that

$$y_{t+1}(e) = \beta(e)\,\varepsilon_{1t+1} + \gamma(e)\,y_t(e) + \varepsilon_{2t+1},$$

which shows that $y_{t+1}(e)$ can be obtained from $y_t(e)$. Replacing $y_t(e) = \beta_{t-1}e + V_{0t}$,

$$y_{t+1}(e) = \gamma(e)\,\beta_{t-1}e + V_{t+1}(e) \equiv m_1(e, U_{t+1}), \tag{1}$$

where

$$V_{t+1}(e) = \gamma(e)\,V_{0t} + \beta(e)\,\varepsilon_{1t+1} + \varepsilon_{2t+1} \equiv V_1(e, U_{t+1})$$

with

$$U_{t+1} = (\varepsilon'_{t+1}, \varepsilon_{2t}, z'_{t-1})' \equiv (\varepsilon'_{t+1}, U'_t).$$

For $h = 2$, writing $\beta_{t+1} \equiv \beta(\varepsilon_{1t+1})$ and $\gamma_{t+1} \equiv \gamma(\varepsilon_{1t+1})$, it follows that

$$\begin{aligned} y_{t+2}(e) &= \beta_{t+1}\varepsilon_{1t+2} + \gamma_{t+1}y_{t+1}(e) + \varepsilon_{2t+2} \\ &= \beta_{t+1}\varepsilon_{1t+2} + \gamma_{t+1}[\gamma(e)\,\beta_{t-1}e + V_{t+1}(e)] + \varepsilon_{2t+2} \\ &= \gamma_{t+1}\gamma(e)\,\beta_{t-1}e + V_{t+2}(e) \equiv m_2(e, U_{t+1}), \end{aligned}$$

where

$$\begin{aligned} V_{t+2}(e) &\equiv \gamma_{t+1}V_{t+1}(e) + \beta_{t+1}\varepsilon_{1t+2} + \varepsilon_{2t+2} \\ &= \gamma_{t+1}[\gamma(e)\,V_{0t} + \beta(e)\,\varepsilon_{1t+1} + \varepsilon_{2t+1}] + \beta_{t+1}\varepsilon_{1t+2} + \varepsilon_{2t+2} \\ &= \gamma_{t+1}\gamma(e)\,V_{0t} + \gamma_{t+1}\beta(e)\,\varepsilon_{1t+1} + \varepsilon_{2t+1} + \beta_{t+1}\varepsilon_{1t+2} + \varepsilon_{2t+2}, \end{aligned}$$

which is a function of $U_{t+2} \equiv (\varepsilon'_{t+2}, \varepsilon'_{t+1}, \varepsilon_{2t}, z'_{t-1})' = (\varepsilon'_{t+2}, U'_{t+1})'$. For any $h > 1$,

$$y_{t+h}(e) = \gamma_{t+h-1}\cdots\gamma_{t+1}\gamma(e)\,\beta_{t-1}e + V_{t+h}(e) \equiv m_h(e, U_{t+h}),$$

where

$$V_{t+h}(e) \equiv \gamma_{t+h-1} V_{t+h-1}(e) + \beta_{t+h-1}\varepsilon_{1t+h} + \varepsilon_{2t+h},$$

and $U_{t+h} \equiv \left(\varepsilon'_{t+h}, U'_{t+h-1}\right)'$.

Next, we show part (i) of the proposition, which derives the conditional average response function for any fixed δ. For $h = 0$, $y_t(e+\delta) - y_t(e) = \beta_{t-1}\delta$, which does not depend on e. Hence,

$$CAR_0\left(\delta, \bar{h}\right) = E\left(y_t(\varepsilon_{1t}+\delta) - y_t(\varepsilon_{1t}) | H_{t-1} = \bar{h}\right) = E\left(\beta_{t-1} | H_{t-1} = \bar{h}\right)\delta = \beta_{\bar{h}}\delta.$$

For $h = 1$, by Definition 1,

$$CAR_1\left(\delta, \bar{h}\right) = E\left(y_{t+1}(\varepsilon_{1t}+\delta) - y_{t+1}(\varepsilon_{1t}) | H_{t-1} = \bar{h}\right),$$

where $y_{t+1}(\varepsilon_{1t})$ is equal to $y_{t+1}(e)$ with $e = \varepsilon_{1t}$ (and similarly for $y_{t+1}(\varepsilon_{1t}+\delta)$). We will evaluate $CAR_1\left(\delta, \bar{h}\right)$ below, but note that under the simplified Assumption 3, for any $h > 1$, we can write $CAR_h\left(\delta, \bar{h}\right)$ as a function of $CAR_1\left(\delta, \bar{h}\right)$. Specifically, for $h = 2$, we have that

$$\begin{aligned} y_{t+2}(e+\delta) - y_{t+2}(e) &= \gamma_{t+1} y_{t+1}(e+\delta) + \beta_{t+1}\varepsilon_{1t+2} + \varepsilon_{2t+2} - \left(\gamma_{t+1} y_{t+1}(e) + \beta_{t+1}\varepsilon_{1t+2} + \varepsilon_{2t+2}\right) \\ &= \gamma_{t+1}\left[y_{t+1}(e+\delta) - y_{t+1}(e)\right], \end{aligned}$$

and more generally for any $h > 1$,

$$y_{t+h}(e+\delta) - y_{t+h}(e) = \gamma_{t+h-1}\left[y_{t+h-1}(e+\delta) - y_{t+h-1}(e)\right] = \left(\gamma_{t+h-1}\cdots\gamma_{t+1}\right)\left[y_{t+1}(e+\delta) - y_{t+1}(e)\right].$$

By Definition 1, for any $h > 1$,

$$\begin{aligned} CAR_h\left(\delta, \bar{h}\right) &= E\left[y_{t+h}(\varepsilon_{1t}+\delta) - y_{t+h}(\varepsilon_{1t}) | H_{t-1} = \bar{h}\right] \\ &= E\left(\gamma_{t+h-1}\cdots\gamma_{t+1}\right) E\left[y_{t+1}(\varepsilon_{1t}+\delta) - y_{t+1}(\varepsilon_{1t}) | H_{t-1} = \bar{h}\right] \\ &= \left(\bar{\gamma}\right)^{h-1} CAR_1\left(\delta, \bar{h}\right), \end{aligned} \tag{2}$$

where we let $\bar{\gamma} \equiv E(\gamma_{t+1})$ for any t. The last equality follows from the fact that γ_t is a function of ε_{1t} and ε_{1t} is i.i.d. This implies that we only need to evaluate $CAR_1\left(\delta, \bar{h}\right)$ and $\bar{\gamma}$ to obtain the entire conditional average response function. Under Assumption 3(a) and (b), where the Gaussianity assumption is instrumental in deriving the closed form expressions for $\bar{\gamma}$ and $CAR_1\left(\delta, \bar{h}\right)$, using (1), for any fixed e,

$$
\begin{aligned}
y_{t+1}(e+\delta) - y_{t+1}(e) &= \gamma(e)\beta_{t-1}\delta \\
&\quad +[\gamma(e+\delta) - \gamma(e)]\beta_{t-1}\delta \\
&\quad +[\gamma(e+\delta) - \gamma(e)]\beta_{t-1}e \\
&\quad +[\gamma(e+\delta) - \gamma(e)]V_{0t} \\
&\quad +[\beta(e+\delta) - \beta(e)]\varepsilon_{1t+1}.
\end{aligned}
$$

Next, evaluate this difference at $e = \varepsilon_{1t}$ and take the expectation, conditionally on $H_{t-1} = \bar{h}$. It follows that for any fixed δ,

$$
\begin{aligned}
CAR_1\left(\delta, \bar{h}\right) &\equiv E\left[y_{t+1}(\varepsilon_{1t}+\delta) - y_{t+1}(\varepsilon_{1t})|H_{t-1} = \bar{h}\right] \\
&= E\left[\gamma(\varepsilon_{1t})|H_{t-1} = \bar{h}\right]\beta_{\bar{h}}\delta + \{E\left[(\gamma(\varepsilon_{1t}+\delta) - \gamma(\varepsilon_{1t}))|H_{t-1} = \bar{h}\right]\beta_{\bar{h}}\delta \\
&\quad +E\left[(\gamma(\varepsilon_{1t}+\delta) - \gamma(\varepsilon_{1t}))\varepsilon_{1t}|H_{t-1} = \bar{h}\right]\beta_{\bar{h}} + E\left[(\gamma(\varepsilon_{1t}+\delta) - \gamma(\varepsilon_{1t}))V_{0t}|H_{t-1} = \bar{h}\right] \\
&\quad +E[(\beta(\varepsilon_{1t}+\delta) - \beta(\varepsilon_{1t}))\varepsilon_{1t+1}|H_{t-1} = \bar{h}]\} \qquad (3)
\end{aligned}
$$

Note that the last term in (3) has conditional mean zero. This follows by the law of iterated expectations, using the fact that ε_{1t} is an i.i.d. zero mean random variable which is independent of ε_{2t}. Under these assumptions, V_{0t} is independent of ε_{1t}, and the second-to-last term can be written as $E(\gamma(\varepsilon_{1t}+\delta) - \gamma(\varepsilon_{1t}))]v_{\bar{h}}$ (where $v_{\bar{h}} = E\left(V_{0t}|H_{t-1} = \bar{h}\right) = \gamma_{\bar{h}}E(y_{t-1}|H_{t-1} = \bar{h})$). By using similar arguments, we can decompose $CAR_1\left(\delta, \bar{h}\right)$ into the sum of

$$
\begin{aligned}
\text{Direct effect} &= E(\gamma(\varepsilon_{1t}))\beta_{\bar{h}}\delta. \\
\text{Indirect effect} &= E\left[(\gamma(\varepsilon_{1t}+\delta) - \gamma(\varepsilon_{1t}))\right]\beta_{\bar{h}}\delta \\
&\quad +E\left[(\gamma(\varepsilon_{1t}+\delta) - \gamma(\varepsilon_{1t}))\varepsilon_{1t}\right]\beta_{\bar{h}} \\
&\quad +E\left[\gamma(\varepsilon_{1t}+\delta) - \gamma(\varepsilon_{1t})\right]v_{\bar{h}}.
\end{aligned}
$$

This decomposition shows that the first component of $CAR_1\left(\delta, \bar{h}\right)$ captures the direct effect of a shock of size δ in ε_{1t} on y_{t+h}. Since $\gamma(\varepsilon_{1t}) = \gamma_t$, this is the effect of a change in ε_{1t} on y_{t+h} that keeps γ_t constant, as when H_t is exogenous. However, in the current model, $H_t = \eta(\varepsilon_{1t})$, which means that when we perturb ε_{1t} by δ, this also impacts the model parameters at time t. The last three terms in $CAR_1\left(\delta, \bar{h}\right)$ capture this "indirect effect" since they depend on the wedge between $\gamma(\varepsilon_{1t}+\delta)$ and $\gamma(\varepsilon_{1t})$.

Suppose now that $\varepsilon_{1t} \sim N\left(0, \sigma_1^2\right)$, as in Assumption 3(b). Then,

$$E\left(\eta\left(\varepsilon_{1t}+\delta\right)\right)=E\left(1\left(\varepsilon_{1t}+\delta>c\right)\right)=P\left(\varepsilon_{1t}/\sigma_1>\left(c-\delta\right)/\sigma_1\right)=1-\Phi\left(\left(c-\delta\right)/\sigma_1\right)=\Phi\left(-c/\sigma_1+\delta/\sigma_1\right).$$

and

$$E\left(\gamma\left(\varepsilon_{1t}+\delta\right)\right)=\gamma_R+\left(\gamma_E-\gamma_R\right)\Phi\left(-c/\sigma_1+\delta/\sigma_1\right).$$

Also, we can show that

$$\begin{aligned}
E\left[\left(\gamma\left(\varepsilon_{1t}+\delta\right)-\gamma\left(\varepsilon_{1t}\right)\right)\varepsilon_{1t}\right] &= \left(\gamma_E-\gamma_R\right)E\left[\left(\eta\left(\varepsilon_{1t}+\delta\right)-\eta\left(\varepsilon_{1t}\right)\right)\varepsilon_{1t}\right] \\
&= \left(\gamma_E-\gamma_R\right)E\left[\left(1\left(\varepsilon_{1t}+\delta>c\right)-1\left(\varepsilon_{1t}>c\right)\right)\varepsilon_{1t}\right] \\
&= \left(\gamma_E-\gamma_R\right)E\left[\left(1\left(\left(c-\delta\right)/\sigma_1<\varepsilon_{1t}/\sigma_1<c/\sigma_1\right)\right)\frac{\varepsilon_{1t}}{\sigma_1}\right]\sigma_1 \\
&= \left(\gamma_E-\gamma_R\right)\sigma_1\left[\phi\left(\left(c-\delta\right)/\sigma_1\right)-\phi\left(c/\sigma_1\right)\right] \\
&= \left(\gamma_E-\gamma_R\right)\sigma_1\left[\phi\left(-c/\sigma_1+\delta/\sigma_1\right)-\phi\left(-c/\sigma_1\right)\right].
\end{aligned}$$

It follows that

$$\begin{aligned}
CAR_1\left(\delta,\bar{h}\right) &= E\left[\gamma\left(\varepsilon_{1t}+\delta\right)\right]\beta_{\bar{h}}\delta+E\left[\left(\gamma\left(\varepsilon_{1t}+\delta\right)-\gamma\left(\varepsilon_{1t}\right)\right)\varepsilon_{1t}\right]\beta_{\bar{h}}-E\left[\gamma\left(\varepsilon_{1t}+\delta\right)-\gamma\left(\varepsilon_{1t}\right)\right]v_{\bar{h}} \\
&= \left\{\gamma_R+\left(\gamma_E-\gamma_R\right)\Phi(-c/\sigma_1+\delta/\sigma_1)\right\}\beta_{\bar{h}}\delta+\left(\gamma_E-\gamma_R\right)\sigma_1[\phi\left(-c/\sigma_1+\delta/\sigma_1\right)-\phi\left(-c/\sigma_1\right))]\beta_{\bar{h}} \\
&\quad +\{\left(\gamma_E-\gamma_R\right)[\Phi\left(-c/\sigma_1+\delta/\sigma\right)-\Phi\left(-c/\sigma_1\right)]v_{\bar{h}}\} \\
&= \underbrace{\left\{\gamma_R+\left(\gamma_E-\gamma_R\right)\Phi(-c/\sigma_1)\right\}\beta_{\bar{h}}\delta}_{=E(\gamma_t)\beta_{\bar{h}}\delta=\text{Direct effect}} \\
&\quad +\left\{\gamma_R+\left(\gamma_E-\gamma_R\right)\left[\Phi(-c/\sigma_1+\delta/\sigma_1)-\Phi(-c/\sigma_1)\right]\right\}\beta_{\bar{h}}\delta \\
&\quad +\left\{\left(\gamma_E-\gamma_R\right)\sigma_1(\phi(-c/\sigma_1+\delta/\sigma_1)-\phi(-c/\sigma_1))\right\}\beta_{\bar{h}} \qquad (4) \\
&\quad +\left\{\left(\gamma_E-\gamma_R\right)\left[\Phi\left(-c/\sigma_1+\delta/\sigma_1\right)-\Phi\left(-c/\sigma_1\right)\right]\right\}v_{\bar{h}},
\end{aligned}$$

where the last three terms define the "Indirect effect". Plugging this expression into (2) gives the formula for $CAR_h\left(\delta,\bar{h}\right)$ for any $h>1$ and any fixed δ. Note that

$$\bar{\gamma}=E\left(\gamma_t\right)=\gamma_R+\left(\gamma_E-\gamma_R\right)\Phi(-c/\sigma_1)\text{ for all }t.$$

To prove part (ii), we use the fact that

$$\begin{aligned}
CMR_h\left(\bar{h}\right) &= \lim_{\delta\to 0}[\delta^{-1}CAR_h\left(\delta,\bar{h}\right)] \\
&= \left(\bar{\gamma}\right)^{h-1}\lim_{\delta\to 0}[\delta^{-1}CAR_1\left(\delta,\bar{h}\right)] \\
&= \left(\bar{\gamma}\right)^{h-1}CMR_1\left(\bar{h}\right),
\end{aligned}$$

where $CMR_1\left(\bar{h}\right) = \lim_{\delta\to 0} CAR_1\left(\delta, \bar{h}\right)/\delta$. In particular, by dividing (4) by δ and taking the limit as $\delta \to 0$, we get

$$CMR_1\left(\bar{h}\right) = \left\{\gamma_R + \left(\gamma_E - \gamma_R\right)\Phi(-c/\sigma_1)\right\}\beta_{\bar{h}} + I_0 + I_1 + I_2,$$

where

$$\begin{aligned}
I_0 &= \lim_{\delta\to 0}\delta^{-1}\left\{\gamma_R + \left(\gamma_E - \gamma_R\right)\left[\Phi(-c/\sigma_1 + \delta/\sigma_1) - \Phi(-c/\sigma_1)\right]\right\}\beta_{\bar{h}}\delta = 0 \\
I_1 &= \lim_{\delta\to 0}\delta^{-1}\left\{\left(\gamma_E - \gamma_R\right)\sigma_1(\phi(-c/\sigma_1 + \delta/\sigma_1) - \phi(-c/\sigma_1))\right\}\beta_{\bar{h}} \\
I_2 &= \lim_{\delta\to 0}[\delta^{-1}\left(\gamma_E - \gamma_R\right)\left[\Phi\left(-c/\sigma_1 + \delta/\sigma_1\right) - \Phi\left(-c/\sigma_1\right)\right]]v_{\bar{h}}.
\end{aligned}$$

We can evaluate I_1 and I_2 by using the following two Taylor expansions of the Gaussian pdf and cdf,

$$\begin{aligned}
\phi(-c/\sigma_1 + \delta/\sigma_1) &= \phi(-c/\sigma_1) + \phi'(-c/\sigma_1)\frac{\delta}{\sigma_1} + O\left(\delta^2\right), \\
\Phi(-c/\sigma_1 + \delta/\sigma_1) &= \Phi(-c/\sigma_1) + \Phi'(-c/\sigma_1)\frac{\delta}{\sigma_1} + O\left(\delta^2\right),
\end{aligned}$$

where $\Phi'(-c/\sigma_1) = \phi(-c/\sigma_1) = \phi(c/\sigma_1)$ and $\phi'\left(-c/\sigma_1\right) = -\left(-c/\sigma_1\right)\phi\left(-c/\sigma_1\right) = \phi(c/\sigma_1)c/\sigma_1$ by the properties of the Gaussian pdf and cdf (in particular, note that $\Phi'\left(x\right) = \phi\left(x\right)$, $\phi\left(x\right) = \phi\left(-x\right)$ and $\phi'\left(x\right) = -x\phi\left(x\right)$). Hence,

$$I_1 = \left(\gamma_E - \gamma_R\right)\sigma_1\phi(c/\sigma_1)c/\sigma_1^2\beta_{\bar{h}} = \left(\gamma_E - \gamma_R\right)\phi(c/\sigma_1)c/\sigma_1\beta_{\bar{h}},$$

and

$$I_2 = \left(\gamma_E - \gamma_R\right)\sigma_1^{-1}\phi\left(c/\sigma_1\right)v_{\bar{h}}.$$

Thus,

$$CMR_1\left(\bar{h}\right) = \left\{\gamma_R + \left(\gamma_E - \gamma_R\right)\Phi(-c/\sigma_1)\right\}\beta_{\bar{h}} + \left(\gamma_E - \gamma_R\right)\phi\left(c/\sigma_1\right)\sigma_1^{-1}\left(c\beta_{\bar{h}} + v_{\bar{h}}\right).$$

■

Proof of Proposition 3.4. The result for $h = 0$ is immediate, so we focus on $h \geq 1$. For any such value of h, we can show that

$$b_h\left(\bar{h}\right) = \frac{E\left(y_{t+h}\varepsilon_{1t}|H_{t-1} = \bar{h}\right)}{E\left(\varepsilon_{1t}^2|H_{t-1} = \bar{h}\right)} = \left(\bar{\gamma}\right)^{h-1}b_1\left(\bar{h}\right),$$

using the fact that γ_t is i.i.d. since it is a function of ε_{1t}. Thus, we focus on deriving $b_1\left(\bar{h}\right) = \frac{E\left(y_{t+1}\varepsilon_{1t}|H_{t-1}=\bar{h}\right)}{E\left(\varepsilon_{1t}^2|H_{t-1}=\bar{h}\right)}$. Note that the denominator of $b_1\left(\bar{h}\right)$ is equal to σ_1^2 under our assumptions, so it is sufficient to derive $E\left(y_{t+1}\varepsilon_{1t}|H_{t-1} = \bar{h}\right)$. Replacing y_{t+1} by equation (3) in the main text, we write

$$E\left(y_{t+1}\varepsilon_{1t}|H_{t-1} = \bar{h}\right) = E\left(\left(\beta_t\varepsilon_{1t+1} + \gamma_t y_t + \varepsilon_{2t+1}\right)\varepsilon_{1t}|H_{t-1} = \bar{h}\right) = E(\gamma_t y_t\varepsilon_{1t}|H_{t-1} = \bar{h}),$$

since $E\left(\beta_t\varepsilon_{1t+1}\varepsilon_{1t}|H_{t-1}=\bar{h}\right)=E\left(\varepsilon_{2t+1}\varepsilon_{1t}|H_{t-1}=\bar{h}\right)=0$. But since $\gamma_t=\gamma_R+(\gamma_E-\gamma_R)H_t$,

$$E(\gamma_t y_t\varepsilon_{1t}|H_{t-1}=\bar{h})=(\gamma_E-\gamma_R)E\left(H_t y_t\varepsilon_{1t}|H_{t-1}=\bar{h}\right)+\gamma_R E\left(y_t\varepsilon_{1t}|H_{t-1}=\bar{h}\right)\equiv(\gamma_E-\gamma_R)A_1+\gamma_R A_2.$$

It follows that

$$\begin{aligned}
A_1 &\equiv E\left(\varepsilon_{1t}H_t y_t|H_{t-1}=\bar{h}\right)\\
&= E\left(\varepsilon_{1t}H_t\left(\beta_{t-1}\varepsilon_{1t}+\gamma_{t-1}y_{t-1}+\varepsilon_{2t}\right)|H_{t-1}=\bar{h}\right)\\
&= E\left(\varepsilon_{1t}^2H_t|H_{t-1}=\bar{h}\right)\beta_{\bar{h}}+E\left(\varepsilon_{1t}H_t\gamma_{t-1}y_{t-1}|H_{t-1}=\bar{h}\right)+E\left(\varepsilon_{1t}\varepsilon_{2t}H_t|H_{t-1}=\bar{h}\right)\\
&= E\left(\varepsilon_{1t}^2H_t\right)\beta_{\bar{h}}+E\left(\varepsilon_{1t}H_t\right)\underbrace{E\left(\gamma_{t-1}y_{t-1}|H_{t-1}=\bar{h}\right)}_{\equiv v_{\bar{h}}}+0,
\end{aligned}$$

where $E\left(\varepsilon_{1t}\varepsilon_{2t}H_t|H_{t-1}=\bar{h}\right)=0$ by the fact that $\varepsilon_{1t}H_t$ is independent of ε_{2t} under Assumptions 1 and 3. Similarly, we can write $E\left(\varepsilon_{1t}H_t\gamma_{t-1}y_{t-1}|H_{t-1}=\bar{h}\right)=E\left(\varepsilon_{1t}H_t\right)v_{\bar{h}}$, where $v_{\bar{h}}\equiv E\left(V_{0t}|H_{t-1}=\bar{h}\right)=E(\gamma_{t-1}y_{t-1}|H_{t-1}=\bar{h})$. Next, we compute $E\left(\varepsilon_{1t}H_t\right)$ and $E\left(\varepsilon_{1t}^2H_t\right)$ using the fact that ε_{1t} is Gaussian. By definition of $H_t=1\left(\varepsilon_{1t}>c\right)$, and the truncated moments of the Gaussian distribution, we obtain that

$$E\left(\varepsilon_{1t}H_t\right)=\sigma_1E\left(\varepsilon_{1t}/\sigma_1 1\left(\varepsilon_{1t}/\sigma_1>c/\sigma_1\right)\right)=\sigma_1\phi\left(c/\sigma_1\right).$$

Similarly,

$$E\left(\varepsilon_{1t}^2H_t\right)=E\left(\varepsilon_{1t}^2 1\left(\varepsilon_{1t}>c\right)\right)=\sigma_1^2[\Phi\left(-c/\sigma_1\right)+c/\sigma_1\phi\left(c/\sigma_1\right)].$$

Thus

$$\frac{A_1}{\sigma_1^2}=[\Phi\left(-c/\sigma_1\right)+c/\sigma_1\phi\left(c/\sigma_1\right)]\beta_{\bar{h}}+\sigma_1^{-1}\phi\left(c/\sigma_1\right)v_{\bar{h}}.$$

Since we can also show that

$$\frac{A_2}{\sigma_1^2}=\sigma_1^{-2}E\left(y_t\varepsilon_{1t}|H_{t-1}=\bar{h}\right)=\sigma_1^{-2}E\left((\beta_{t-1}\varepsilon_{1t}+\gamma_{t-1}y_{t-1}+\varepsilon_{2t})\varepsilon_{1t}|H_{t-1}=\bar{h}\right)=\beta_{\bar{h}},$$

it follows that

$$\begin{aligned}
b_1\left(\bar{h}\right) &= (\gamma_E-\gamma_R)\frac{A_1}{\sigma_1^2}+\gamma_R\frac{A_2}{\sigma_1^2}\\
&= (\gamma_E-\gamma_R)\{[\Phi\left(-c/\sigma_1\right)+c/\sigma_1\phi\left(c/\sigma_1\right)]\beta_{\bar{h}}+\sigma_1^{-1}\phi\left(c/\sigma_1\right)v_{\bar{h}}\}+\gamma_R\beta_{\bar{h}}\\
&= \{\gamma_R\beta_{\bar{h}}+(\gamma_E-\gamma_R)\Phi\left(-c/\sigma_1\right)\}\beta_{\bar{h}}+(\gamma_E-\gamma_R)\sigma_1^{-1}\phi\left(c/\sigma_1\right)\left(c\beta_{\bar{h}}+v_{\bar{h}}\right)\\
&= CMR_1\left(\bar{h}\right).
\end{aligned}$$

■

B Generalization of Propositions 3.1 and 3.2

Here, we show that the results in Section 3.1 extend to a multivariate version of our model for $z_t = (x_t, y_t')'$ when H_t is exogenous.

B.1 Multivariate state-dependent structural VAR model

Let $z_t \equiv (x_t, y_t')'$ denote an $n \times 1$ vector of strictly stationary time series, where y_t is $k \times 1$ with $k = n - 1$. We consider a structural state-dependent VAR process of the form

$$C_{t-1} z_t = \mu_{t-1} + B_{t-1}(L) z_{t-1} + \varepsilon_t, \tag{5}$$

where $\varepsilon_t = (\varepsilon_{1t}, \varepsilon_{2t}')'$ defines the vector of mutually independent structural shocks. Let

$$B_{t-1}(L) = B_{1,t-1} + B_{2,t-1} L + \ldots + B_{p,t-1} L^{p-1},$$

where p denotes the polynomial lag order. For later convenience, we partition $B_{t-1}(L)$ conformably with z_t as

$$B_{t-1}(L) = \begin{pmatrix} B_{11,t-1}(L) & B_{12,t-1}(L) \\ B_{21,t-1}(L) & B_{22,t-1}(L) \end{pmatrix}$$

where $\mathcal{A}_{ij}$ denotes the (i, j) block of any partitioned matrix $\mathcal{A}$.

All model coefficients evolve over time depending on the state of the economy. In particular, as in the main text, we let

$$\begin{aligned} \mu_{t-1} &= \mu_E H_{t-1} + \mu_R (1 - H_{t-1}), \\ C_{t-1} &= C_E H_{t-1} + C_R (1 - H_{t-1}), \text{ and} \\ B_{j,t-1} &= B_{jE} H_{t-1} + B_{jR} (1 - H_{t-1}) \text{ for } j = 1, \ldots, p, \end{aligned}$$

where H_{t-1} is a binary stationary time series that takes the value 1 if the economy is in expansion and 0 otherwise. To identify the conditional impulse response function of y_{t+h} to a shock in ε_{1t}, we assume that

$$C_{t-1} = \begin{pmatrix} 1 & 0 \\ -C_{21,t-1} & C_{22,t-1} \end{pmatrix}, \tag{6}$$

where $C_{21,t-1}$ is $k \times 1$ and $C_{22,t-1}$ is a $k \times k$ non-singular matrix whose diagonal elements are 1 by a standard normalization condition. Under these assumptions, x_t is predetermined with respect to y_t. Note that we do not restrict $C_{22,t-1}$ to be lower triangular, which allows C_{t-1} to be block recursive. Hence, the model is only partially identified in that only the responses to ε_{1t} are identified.

Model (5) covers several empirically relevant strategies for identifying the structural shock ε_{1t} (and the corresponding conditional response function for y_{t+h} with respect to ε_{1t}). One is the narrative approach to identification which uses information extraneous to the model to measure ε_{1t}, in which case $x_t = \varepsilon_{1t}$ (as in the main text). Alternatively, the structural shock ε_{1t} may be identified via an exclusion restriction that precludes x_t from responding contemporaneously to the structural shocks in the remaining variables of the system. In this case, the structural shock ε_{1t} is identified within the nonlinear structural VAR model by analogy to Blanchard and Perotti (2002), whose exogenous shocks to government spending (ε_{1t}) are identified by assuming that government spending (x_t) does not react within the period to shocks to output and tax revenues (y_t). Finally, note that our general model also accommodates the special case of x_t being an exogenous serially correlated observable variable, as in Alloza, Gonzalo and Sanz (2021).

The structural model for z_t can be written as

$$\begin{cases} x_t = \mu_{1,t-1} + B_{11,t-1}(L)\,x_{t-1} + B_{12,t-1}(L)\,y_{t-1} + \varepsilon_{1t} \\ \\ C_{22,t-1}y_t = \mu_{2,t-1} + C_{21,t-1}x_t + B_{21,t-1}(L)\,x_{t-1} + B_{22,t-1}(L)\,y_{t-1} + \varepsilon_{2t}. \end{cases} \tag{7}$$

Without further restrictions (such as postulating that $C_{22,t-1}$ is lower triangular), the parameters in the equations for y_t are not identified. However, the fact that ε_{1t} is identified suffices to identify the conditional response function of y_t to a one-time shock in ε_{1t}.

As in Section 3.1, we assume that H_{t-1} is a function only of q_t (and its lags), where q_t is assumed to be exogenous with respect to the structural shocks ε_{1t} and ε_{2t}. More specifically, to complete the model, we let

$$H_t = \eta\,(q_s : s \leq t)\,. \tag{8}$$

We make the following additional assumptions.

Assumption B.1 *$\{\varepsilon_{1t}\}$ and $\{\varepsilon_{2t}\}$ are mutually independent structural shocks such that $\varepsilon_t \equiv (\varepsilon_{1t}, \varepsilon_{2t}')' \sim$ i.i.d.$(0, \Sigma)$, where Σ is a diagonal matrix with diagonal elements given by σ_i^2 for $i = 1, \ldots, n$. In addition, y_t is strictly stationary and ergodic.*

Assumption B.2 *$\{q_t\}$ is independent of $\{\varepsilon_{1t}\}$ and $\{\varepsilon_{2t}\}$.*

Assumption B.1 is the generalization of Assumption 1 in Section 3.1 to the multivariate model where ε_{2t} is a $k \times 1$ vector. Assumption B.2 is the analogue of Assumption 2.

B.2 Conditional impulse response functions

In this section, we derive the analogue of Proposition 3.1 in the main text for the multivariate model considered in (7) and (8). We obtain this result by first deriving the potential outcomes $y_{t+h}(e)$ and then using these to obtain closed-form expressions for $CAR_h\left(\delta,\bar{h}\right)$ and $CMR_h\left(\delta,\bar{h}\right)$.

B.2.1 Potential outcomes

To derive the potential outcomes $y_{t+h}(e)$, we first obtain the reduced-form model corresponding to our structural model (7) (which is given by (5) with the identification restriction that x_t is predetermined with respect to ε_{1t}). Since C_{t-1} satisfies the identification condition (6), the inverse matrix of C_{t-1} exists and is given by

$$C_{t-1}^{-1} = \begin{pmatrix} 1 & 0 \\ C_{22,t-1}^{-1}C_{21,t-1} & C_{22,t-1}^{-1} \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ C_{t-1}^{21} & C_{t-1}^{22} \end{pmatrix},$$

where for any matrix $\mathcal{A}$, we let $\mathcal{A}^{ij}$ denote the block (i,j) of $\mathcal{A}^{-1}$.

Pre-multiplying (5) by C_{t-1}^{-1} yields

$$z_t = C_{t-1}^{-1}\mu_{t-1} + C_{t-1}^{-1}B_{t-1}(L)\,z_{t-1} + C_{t-1}^{-1}\varepsilon_t,$$

which we rewrite as

$$z_t = b_{t-1} + A_{t-1}(L)\,z_{t-1} + \eta_t, \tag{9}$$

where $\eta_t \equiv C_{t-1}^{-1}\varepsilon_t$, $b_{t-1} \equiv C_{t-1}^{-1}\mu_{t-1}$, and

$$A_{t-1}(L) \equiv C_{t-1}^{-1}B_{t-1}(L) = A_{1,t-1} + A_{2,t-1}L + \ldots + A_{p,t-1}L^{p-1},$$

with $A_{j,t-1} \equiv C_{t-1}^{-1}B_{j,t-1}$.

The potential outcome value of $y_{t+h}(e)$ (for any fixed e) can be obtained from the companion-form representation of the reduced-form model (9) by iteration, fixing $\varepsilon_{1t}=e$. Since only ε_{1t} is fixed at e, the following decomposition of the reduced-form errors η_t is useful:

$$\eta_t \equiv C_{t-1}^{-1}\varepsilon_t = \begin{pmatrix} 1 \\ C_{t-1}^{21} \end{pmatrix}\varepsilon_{1t} + \begin{pmatrix} 0 \\ C_{t-1}^{22} \end{pmatrix}\varepsilon_{2t} \equiv C_{t-1}^{-1}e_{1,n}\varepsilon_{1t} + C_{t-1}^{-1}I_{2:n}\varepsilon_{2t},$$

where $e_{1,n} \equiv (1,0')'$ is $n\times 1$ and $I_{2:n}$ is $k\times n$ and is equal to the $n\times n$ identity matrix with its first column removed:

$$I_{2:n} = \begin{pmatrix} e_{2,n} & \cdots & e_{n,n} \end{pmatrix}.$$

We let

$$\eta_t(e) = C_{t-1}^{-1}\begin{pmatrix} e \\ \varepsilon_{2t} \end{pmatrix} = C_{t-1}^{-1}e_{1,n}e + C_{t-1}^{-1}I_{2:n}\varepsilon_{2t}$$

denote the counterfactual value of η_t for $\varepsilon_{1t} = e$. Similarly, we denote by

$$z_t(e) = \begin{pmatrix} x_t(e) \\ y_t(e) \end{pmatrix}$$

the counterfactual values of x_t and y_t. With this notation, we can write the potential outcome analogue of (9) as

$$Z_t(e) = a_{t-1} + A_{t-1} Z_{t-1}(e) + \xi_t(e). \tag{10}$$

Here,

$$\underset{np\times 1}{Z_t}(e) = \left(z_t'(e), z_{t-1}'(e), \ldots, z_{t-p+1}'(e)\right)', \ \underset{np\times 1}{\xi_t}(e) = \left(\eta_t'(e), 0'\right)', \ \underset{np\times 1}{a_{t-1}} = \left(b_{t-1}', 0'\right)',$$

and

$$\underset{np\times np}{A_{t-1}} = \begin{pmatrix} A_{1,t-1} & A_{2,t-1} & \cdots & A_{p-1,t-1} & A_{p,t-1} \\ I_n & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & I_n & 0 \end{pmatrix}.$$

Note that a_{t-1} and A_{t-1} are not indexed by e because these matrices depend only on H_{t-1}, which does not change with e under the exogeneity assumption on H_t. To obtain $y_t(e)$ from $Z_t(e)$, let

$$\underset{k\times np}{\mathbb{S}_k} = \begin{pmatrix} 0_{k\times 1} & I_k & 0_{k\times n(p-1)} \end{pmatrix}$$

denote a $k \times np$ selection matrix (with $k = n - 1$ equal to the number of variables in y_t) which selects the subvector y_t from the vector Z_t. With this notation,

$$y_t(e) = \mathbb{S}_k Z_t(e),$$

and, more generally, for any h,

$$y_{t+h}(e) = \mathbb{S}_k Z_{t+h}(e).$$

Note that for $k = 1$ (i.e., for a bivariate system with $n = 2$), $\mathbb{S}_k = e_{2,2p}'$, where $e_{2,2p} = (0, 1, 0')$ is a $2p \times 1$ vector whose only non-zero element is equal to 1 and occurs in position 2. More generally, we let $e_{j,m}$ denote an $m \times 1$ vector with 1 in position j and 0 elsewhere.

Next, we use the companion form (10) to obtain $y_{t+h}(e)$ for different values of h. Starting with $h = 0$, we set $Z_{t-1}(e) = Z_{t-1}$ since Z_{t-1} depends on values of z_t that occur prior to the shock in ε_{1t}. Hence, these values do not depend on e and it follows that

$$y_t(e) = \mathbb{S}_k Z_t(e) = \mathbb{S}_k a_{t-1} + \mathbb{S}_k A_{t-1} Z_{t-1} + \mathbb{S}_k \xi_t(e).$$

By the definition of $\xi_t(e)$, we can write

$$\xi_t(e) = \begin{pmatrix} \eta_t(e) \\ 0 \end{pmatrix} = \begin{pmatrix} C_{t-1}^{-1} e_{1,n} e + C_{t-1}^{-1} I_{2:n} \varepsilon_{2t} \\ 0_{n(p-1)\times 1} \end{pmatrix} = e_{1,p} \otimes \left(C_{t-1}^{-1} e_{1,n} e + C_{t-1}^{-1} I_{2:n} \varepsilon_{2t} \right).$$

Hence,

$$\begin{aligned} \mathbb{S}_k \xi_t(e) &= \mathbb{S}_k [e_{1,p} \otimes \left(C_{t-1}^{-1} e_{1,n} e + C_{t-1}^{-1} I_{2:n} \varepsilon_{2t} \right)] \\ &= \mathbb{S}_k [e_{1,p} \otimes \left(C_{t-1}^{-1} e_{1,n} \right) e] + \mathbb{S}_k [e_{1,p} \otimes (C_{t-1}^{-1} I_{2:n}) \varepsilon_{2t}]. \end{aligned}$$

This implies that

$$y_t(e) = \mathbb{S}_k [e_{1,p} \otimes \left(C_{t-1}^{-1} e_{1,n} \right)] e + V_t,$$

where $V_t \equiv \mathbb{S}_k a_{t-1} + \mathbb{S}_k A_{t-1} Z_{t-1} + \mathbb{S}_k [e_{1,p} \otimes (C_{t-1}^{-1} I_{2:n}) \varepsilon_{2t}]$ is a function of $U_t \equiv \left(\varepsilon_{2t}', q_{t-1}, Z_{t-1}' \right)$. We can obtain $y_{t+h}(e)$ for larger values of h using a similar approach. In particular, for $h = 1$, we have that

$$Z_{t+1}(e) = a_t + A_t Z_t(e) + \xi_{t+1},$$

where $\xi_{t+1} = \left(\eta_{t+1}', 0' \right)' = \left(\left(C_t^{-1} \varepsilon_{t+1} \right)', 0' \right)'$ and a_t, A_t and C_t do not depend on e. This is true because the model coefficients depend on H_t, which is not a function of e when H_t is exogenous, and ε_{t+1} is independent of e since e is the fixed value of ε_{1t}. Thus,

$$\begin{aligned} y_{t+1}(e) &= \mathbb{S}_k Z_{t+1}(e) \\ &= \mathbb{S}_k a_t + \mathbb{S}_k A_t Z_t(e) + \mathbb{S}_k \xi_{t+1} \\ &= \mathbb{S}_k a_t + \mathbb{S}_k A_t (a_{t-1} + A_{t-1} Z_{t-1} + \xi_t(e)) + \mathbb{S}_k \xi_{t+1} \\ &= \mathbb{S}_k a_t + \mathbb{S}_k A_t a_{t-1} + \mathbb{S}_k A_t A_{t-1} Z_{t-1} + \mathbb{S}_k A_t \xi_t(e) + \mathbb{S}_k \xi_{t+1}, \end{aligned}$$

where $\xi_t(e) = [e_{1,p} \otimes \left(C_{t-1}^{-1} e_{1,n} \right)] e + \mathbb{S}_k [e_{1,p} \otimes (C_{t-1}^{-1} I_{2:n}) \varepsilon_{2t}]$. Inserting $\xi_t(e)$ into the equation above and collecting the terms that not depend on e into V_{t+1} yields

$$y_{t+1}(e) = \mathbb{S}_k A_t [e_{1,p} \otimes \left(C_{t-1}^{-1} e_{1,n} \right)] e + V_{t+1},$$

where V_{t+1} is a function of $U_{t+1} \equiv \left(\varepsilon_{t+1}, \varepsilon_{2t}', q_t, q_{t-1}, Z_{t-1}' \right)'$. This result shows that the potential outcome value $y_{t+1}(e)$ is linear in e, as in the main text. This result generalizes to any value of $h \geq 1$ as follows:

$$y_{t+h}(e) = \mathbb{S}_k A_{t+h-1} \cdots A_t [e_{1,p} \otimes \left(C_{t-1}^{-1} e_{1,n} \right)] e + V_{t+h} \equiv m_h(e, U_{t+h}), \tag{11}$$

where V_{t+h} depends on $U_{t+h} \equiv \left(\varepsilon_{t+h}, \ldots, \varepsilon_{t+1}, \varepsilon_{2t}', q_{t+h-1}, \ldots, q_t, q_{t-1}, Z_{t-1}' \right)'$.

Equation (11) defines the potential outcomes for the vector of dependent variables y_t. It represents a linear function of e under the assumption that $H_t = \eta(q_s : s \leq t)$ and q_s is strictly exogenous with respect to ε_{1t} and ε_{2t}.

B.2.2 Closed-form expressions for the conditional response functions

Next, we use (11) to generalize Proposition 3.1 to the multivariate state-dependent structural VAR model given in (7). For any e,

$$y_{t+h}(e+\delta) - y_{t+h}(e) = \mathbb{S}_k A_{t+h-1} \cdots A_t[e_{1,p} \otimes \left(C_{t-1}^{-1} e_{1,n}\right)]\delta,$$

which implies that letting $e = \varepsilon_{1t}$, and taking the conditional expectation, conditionally on $H_{t-1} = \bar{h} \in \{0,1\}$,

$$\begin{aligned} CAR_h\left(\delta, \bar{h}\right) &\equiv E\left(y_{t+h}(\varepsilon_{1t}+\delta) - y_{t+h}(\varepsilon_{1t}) | H_{t-1} = \bar{h}\right) \\ &= \mathbb{S}_k E\left(A_{t+h-1} A_{t+h-2} \dots A_t | H_{t-1} = \bar{h}\right)\left(e_{1,p} \otimes C_{\bar{h}}^{-1} e_{1,n}\right)\delta. \end{aligned}$$

We can also use (11) to obtain the conditional marginal response function for this model. Since $y_{t+h}(e)$ is a linear function of e, it follows that

$$y'_{t+h}(e) \equiv \frac{\partial}{\partial e} m_h(e, U_{t+h}) = \mathbb{S}_k A_{t+h-1} \cdots A_t[e_{1,p} \otimes \left(C_{t-1}^{-1} e_{1,n}\right)].$$

This implies that

$$\begin{aligned} CMR_h\left(\bar{h}\right) &\equiv E\left(y'_{t+h}(\varepsilon_{1t}) | H_{t-1} = \bar{h}\right) \\ &= \mathbb{S}_k E\left(A_{t+h-1} A_{t+h-2} \dots A_t | H_{t-1} = \bar{h}\right)\left(e_{1,p} \otimes C_{\bar{h}}^{-1} e_{1,n}\right) \\ &= CAR_h\left(1, \bar{h}\right), \end{aligned}$$

showing that the conditional marginal response function coincides with the conditional average response function $CAR_h\left(\delta, \bar{h}\right)$ for a shock of size $\delta = 1$.

The following proposition summarizes these results and is the analogue of Proposition 3.1 for the multivariate model considered in (7). We let $C_{\bar{h}}^{-1} = C_E^{-1}$ if $\bar{h} = 1$ and $C_{\bar{h}}^{-1} = C_R^{-1}$ if $\bar{h} = 0$.

Proposition B.1 *Assume the structural process is (7) and (8) with $H_t = \eta(q_s : s \leq t)$. Under Assumptions B.1 and B.2 for $\bar{h} \in \{0,1\}$:*

(i) *For any fixed δ, $CAR_0\left(\delta, \bar{h}\right) = \mathbb{S}_k\left(e_{1,p} \otimes C_{\bar{h}}^{-1} e_{1,n}\right)\delta$, and for any $h \geq 1$,*

$$CAR_h\left(\delta, \bar{h}\right) = \mathbb{S}_k E\left(A_{t+h-1} A_{t+h-2} \dots A_t | H_{t-1} = \bar{h}\right)\left(e_{1,p} \otimes C_{\bar{h}}^{-1} e_{1,n}\right)\delta.$$

(ii) *For any $h \geq 0$, $CMR_h\left(\bar{h}\right) = CAR_h\left(\delta, \bar{h}\right)$.*

As in the simpler model considered in the main text, Proposition B.1 shows that when H_t depends only on $\{q_s : s \leq t\}$, i.e., when H_t is exogenous with respect to the structural shocks ε_t, the two definitions of the conditional impulse response function coincide. Next, we show that the state-dependent local projection estimator recovers asymptotically these two notions of conditional impulse response functions when H_t is exogenous.

B.3 Local projections estimands

A state-dependent LP regression is a direct regression of y_{t+h} onto a constant, x_t and Z_{t-1}, each interacted with H_{t-1} and $1-H_{t-1}$. The slope coefficients associated with $x_t H_{t-1}$ are usually interpreted as the CAR of y_{t+h}, conditionally on $H_{t-1} = 1$, whereas the slope coefficients associated with $x_t(1 - H_{t-1})$ are interpreted as the CAR of y_{t+h} when we condition on $H_{t-1} = 0$. The goal of this section is to derive the probability limits of these slope coefficients and show that they equal $CAR_h\left(\delta, \bar{h}\right)$ when $\delta = 1$, which is equal to the $CMR_h\left(\bar{h}\right)$ for $\bar{h} \in \{0, 1\}$.

Let $W_{t-1} \equiv (1, Z'_{t-1})'$ denote an $(np + 1) \times 1$ vector of control variables which include a constant and p lags of z_t. A state-dependent LP for identifying the causal effect on y_{t+h} of a one-time shock in ε_{1t} of size $\delta = 1$ can be written as

$$y_{t+h} = b_h\left(1\right) x_t H_{t-1} + \Pi_{E,h} W_{t-1} H_{t-1} + b_h\left(0\right) x_t (1 - H_{t-1}) + \Pi_{R,h} W_{t-1} (1 - H_{t-1}) + v_{t+h}, \qquad (12)$$

where the $k \times 1$ vectors $b_h\left(1\right)$ and $b_h\left(0\right)$ contain the main parameters of interest. The LP regression for variable $y_{j,t+h}$ is

$$y_{j,t+h} = b_{h,j}\left(1\right) x_t H_{t-1} + \pi'_{E,j,h} W_{t-1} H_{t-1} + b_{h,j}\left(0\right) x_t (1 - H_{t-1}) + \pi'_{R,j,h} W_{t-1} (1 - H_{t-1}) + v_{j,t+h}, \quad (13)$$

where $j = 2, \ldots, n$. The scalar coefficients $b_{h,j}\left(1\right)$ and $b_{h,j}\left(0\right)$ are the $(j - 1)^{th}$ elements of $b_h\left(1\right)$ and $b_h\left(0\right)$, respectively. Similarly, $\pi'_{E,j,h}$ and $\pi'_{R,j,h}$ are the corresponding rows of $\Pi_{E,h}$ and $\Pi_{R,h}$.

Since H_t is observed, the coefficients in the multivariate state-dependent LP regression (12) can be obtained by running a multivariate LS regression of y_{t+h} onto $x_t H_{t-1}$, $W_{t-1} H_{t-1}$, $x_t\left(1 - H_{t-1}\right)$ and $W_{t-1}\left(1 - H_{t-1}\right)$. Note that this is equivalent to running a regression of $y_{j,t+h}$ onto $x_t H_{t-1}$, $W_{t-1} H_{t-1}$, $x_t\left(1 - H_{t-1}\right)$ and $W_{t-1}\left(1 - H_{t-1}\right)$, for each $j = 2, \ldots, n$. Put differently, the multivariate LS regression (12) is equivalent to the k univariate OLS regressions (13), equation-by-equation.

Let $\hat{b}_h\left(1\right)$ and $\hat{b}_h\left(0\right)$ denote the LS estimators of $b_h\left(1\right)$ and $b_h\left(0\right)$ in (12) based on a sample of size T given by $\{y_{t+h}, x_t, Z_{t-1}, H_{t-1} : t = 1, \ldots, T\}$. We can estimate each of these vectors separately, by restricting the sample to $H_{t-1} = 1$ and $H_{t-1} = 0$, respectively. For instance, $\hat{b}_h\left(1\right)$ can be obtained from a regression of y_{t+h} on $x_t H_{t-1}$ and $W_{t-1} H_{t-1}$ (omitting $x_t\left(1 - H_{t-1}\right)$ and $W_{t-1}\left(1 - H_{t-1}\right)$ in the regression). This follows because $H_{t-1}(1-H_{t-1})=0$ for all t. Similarly, we can obtain $\hat{b}_h(0)$ from a regression of y_{t+h} on $x_t(1-H_{t-1})$ and $W_{t-1}(1-H_{t-1})$ (omitting $x_t H_{t-1}$ and $W_{t-1}H_{t-1}$ in this regression).

Our next result generalizes Proposition 3.2. to the multivariate structural VAR model given in (7) and (8).

Proposition B.2 *Consider the structural process (7) and (8) with $H_t = \eta(q_s : s \leq t)$. If Assumptions B.1 and B.2 hold, then for $\bar{h} \in \{0,1\}$,*

$$b_h\left(\bar{h}\right) \equiv p\lim_{T\to\infty} \hat{b}_h\left(\bar{h}\right) = CMR_h\left(\bar{h}\right) = CAR_h\left(1,\bar{h}\right),$$

where $CAR_h\left(1,\bar{h}\right)$ is the conditional average response function in Definition 1 with $\delta = 1$.

B.4 Proofs of Propositions B.1 and B.2

Proof of Proposition B.1. The proof for $h=0$ and $h=1$ is in the text. We omit the proof for general h since it follows from similar arguments.

Proof of Proposition B.2. We focus on $\bar{h}=1$. To define $\hat{b}_h(1)$, let

$$\underset{T\times k}{Y} = \begin{pmatrix} y'_{1+h} \\ \vdots \\ y'_{T+h} \end{pmatrix}, \quad \underset{T\times 1}{X_1} = \begin{pmatrix} x_1 H_0 \\ \vdots \\ x_T H_{T-1} \end{pmatrix}, \quad \text{and} \quad \underset{T\times(np+1)}{X_2} = \begin{pmatrix} W'_0 H_0 \\ \vdots \\ W'_{T-1}H_{T-1} \end{pmatrix},$$

and define $M_2 = I_T - X_2\left(X'_2X_2\right)^{-1}X'_2$.

By the Frisch-Waugh-Lovell (FWL) Theorem, $\hat{b}_h(1)' = \left(X'_1M_2X_1\right)^{-1}X'_1M_2Y$, or

$$\hat{b}_h(1) = T^{-1}(Y'M_2X_1)\left(T^{-1}X'_1M_2X_1\right)^{-1} \equiv \hat{Q}_{1y.2,h}\hat{Q}_{11.2}^{-1}.$$

A similar expression holds for $\hat{b}_h(0)$ with the difference that the regressors x_t and W_{t-1} are interacted with $1-H_{t-1}$ rather than H_{t-1}.

Our goal is to derive the probability limit of $\hat{b}_h(1)$ (and $\hat{b}_h(0)$) as $T\to\infty$. We can write

$$\begin{aligned} \hat{Q}_{11.2} &= T^{-1}X'_1X_1 - T^{-1}X'_1X_2\left(T^{-1}X'_2X_2\right)^{-1}T^{-1}X'_2X_1, \text{ and} \\ \hat{Q}_{1y.2,h} &= T^{-1}Y'X_1 - T^{-1}Y'X_2\left(T^{-1}X'_2X_2\right)^{-1}T^{-1}X'_2X_1. \end{aligned}$$

If a law of large numbers applies to each term[1],

$$\begin{aligned}\hat{Q}_{11.2} \xrightarrow{p} Q_{11.2} &\equiv E\left(x_t^2 H_{t-1}\right) - E\left(x_t H_{t-1} W_{t-1}'\right)\left[E\left(W_{t-1} W_{t-1}' H_{t-1}\right)\right]^{-1} E\left(W_{t-1} H_{t-1} x_t\right), \text{ and} \\ \hat{Q}_{1y.2,h} \xrightarrow{p} Q_{1y.2,h} &\equiv E\left(y_{t+h} x_t H_{t-1}\right) - E\left(y_{t+h} H_{t-1} W_{t-1}'\right)\left[E\left(W_{t-1} W_{t-1}' H_{t-1}\right)\right]^{-1} E\left(W_{t-1} H_{t-1} x_t\right).\end{aligned}$$

We distinguish two cases: (i) $x_t = \varepsilon_{1t}$, and (ii) $x_t = \mu_{1,t-1} + B_{11,t-1}(L)\, x_{t-1} + B_{12,t-1}(L)\, y_{t-1} + \varepsilon_{1t} = \alpha_{t-1}' W_{t-1} + \varepsilon_{1t}$ (where α_{t-1} is a state-dependent vector that collects the coefficients of $\mu_{1,t-1}$, $B_{11,t-1}(L)$ and $B_{12,t-1}(L)$).

In case (i), it is easy to see that $E\left(x_t H_{t-1} W_{t-1}'\right) = 0$ under the assumption that $x_t = \varepsilon_{1t}$ is i.i.d. and independent of ε_{2t}. Thus,

$$Q_{11.2} = E\left(x_t^2 H_{t-1}\right) \text{ and } Q_{1y.2,h} = E\left(y_{t+h} x_t H_{t-1}\right),$$

implying that[2]

$$\hat{b}_h(1) \xrightarrow{p} b_h(1) \equiv E\left(y_{t+h} x_t H_{t-1}\right)\left[E\left(x_t^2 H_{t-1}\right)\right]^{-1} = E\left(y_{t+h} x_t | H_{t-1} = 1\right)\left[E\left(x_t^2 | H_{t-1} = 1\right)\right]^{-1}.$$

In case (ii), we can show that

$$\begin{aligned}Q_{11.2} &= E\left(\varepsilon_{1t}^2 H_{t-1}\right) = \Pr\left(H_{t-1} = 1\right) E\left(\varepsilon_{1t}^2 | H_{t-1} = 1\right) \text{ and} \\ Q_{1y.2,h} &= E\left(y_{t+h} \varepsilon_{1t} H_{t-1}\right) = \Pr\left(H_{t-1} = 1\right) E\left(y_{t+h} \varepsilon_{1t} | H_{t-1} = 1\right),\end{aligned}$$

implying that $\hat{b}_h(1) = E\left(y_{t+h} \varepsilon_{1t} | H_{t-1} = 1\right)\left[E\left(\varepsilon_{1t}^2 | H_{t-1} = 1\right)\right]^{-1}$. Heuristically, this follows because by the FWL theorem, and conditioning on $H_{t-1} = 1$, the slope coefficient associated with x_t from regressing y_{t+h} on x_t and W_{t-1} can be obtained in two steps. First, we regress x_t on W_{t-1} (interacted with H_{t-1}) and obtain the residual. Under our identification condition, this is ε_{1t}. Then, we regress y_{t+h} on ε_{1t} (interacted with H_{t-1}). More specifically, note that

$$E\left(x_t H_{t-1} W_{t-1}'\right) = E\left(\alpha_{t-1}' W_{t-1} W_{t-1}' H_{t-1}\right) + E\left(\varepsilon_{1t} H_{t-1} W_{t-1}'\right) = E\left(\alpha_{t-1}' W_{t-1} W_{t-1}' H_{t-1}\right),$$

[1] This follows under the assumption that z_t is strictly stationary and ergodic and that the usual moment and rank conditions on the regressors are satisfied. We leave these as implicit high level assumptions since our focus here is on the conditions that H_t needs to satisfy in order for the LP estimator to be consistent. Kole and van Dijk (2021) (and references therein) provide primitive conditions for stationarity and ergodicity of a Markov Switching SVAR model when the states H_t are assumed to be a first-order exogenous Markov process. Deriving analogous primitive conditions for our setting, when the process for the exogenous H_t is not specified, is beyond the scope of this paper.

[2] This result is consistent with the fact that when x_t is a directly observed shock we can simply regress y_{t+h} onto $x_t H_{t-1}$ to obtain a consistent estimator of $b_{E,h}$. When $x_t = \varepsilon_{1t}$, adding the controls $W_{t-1} H_{t-1}$ is not required for consistency, but can be important for efficiency.

since $E\left(\varepsilon_{1t} H_{t-1} W'_{t-1}\right) = 0$ by Assumption B.1. It follows that

$$E\left(x_t H_{t-1} W'_{t-1}\right) = \alpha'_E E\left(W_{t-1} W'_{t-1} | H_{t-1} = 1\right) \Pr\left(H_{t-1} = 1\right).$$

Hence, the term $E\left(x_t H_{t-1} W'_{t-1}\right) \left[E\left(W_{t-1} W'_{t-1} H_{t-1}\right)\right]^{-1} E\left(W_{t-1} H_{t-1} x_t\right)$ equals

$$\begin{aligned} & \alpha'_E E\left(W_{t-1} W'_{t-1} | H_{t-1} = 1\right) \left[E\left(W_{t-1} W'_{t-1} | H_{t-1} = 1\right)\right]^{-1} E\left(W_{t-1} W'_{t-1} | H_{t-1} = 1\right) \alpha_E \Pr\left(H_{t-1} = 1\right) \\ = \; & \alpha'_E E\left(W_{t-1} W'_{t-1} | H_{t-1} = 1\right) \alpha_E \Pr\left(H_{t-1} = 1\right) \\ = \; & E\left(\alpha'_{t-1} W_{t-1} W'_{t-1} \alpha_{t-1} | H_{t-1} = 1\right) \Pr\left(H_{t-1} = 1\right). \end{aligned}$$

Since $x_t^2 = \left(\alpha'_{t-1} W_{t-1} + \varepsilon_{1t}\right)^2 = \alpha'_{t-1} W_{t-1} W'_{t-1} \alpha_{t-1} + 2\alpha'_{t-1} W_{t-1} \varepsilon_{1t} + \varepsilon_{1t}^2$, where the second term has a conditional mean of zero, it follows that

$$Q_{11.2} = \Pr\left(H_{t-1} = 1\right) E\left(\varepsilon_{1t}^2 | H_{t-1} = 1\right).$$

One can use similar arguments to show that

$$Q_{1y.2,h} = \Pr\left(H_{t-1} = 1\right) E\left(y_{t+h} \varepsilon_{1t} | H_{t-1} = 1\right).$$

Thus, both in cases (i) and (ii), we conclude that

$$\hat{b}_h(1) \xrightarrow{p} b_h(1) = E\left(y_{t+h} \varepsilon_{1t} | H_{t-1} = 1\right) \left[E\left(\varepsilon_{1t}^2 | H_{t-1} = 1\right)\right]^{-1} \equiv \mathcal{N}_h \mathcal{D},$$

where $\mathcal{N}_h$ stands for numerator and $\mathcal{D}$ is the denominator. Next, we express $\mathcal{N}_h$ and $\mathcal{D}$ in terms of the model parameters. To evaluate $\mathcal{N}_h$, we use the fact that for any h, $y_{t+h} = \mathbb{S}_k Z_{t+h}$, where Z_{t+h} is obtained from the companion-form representation of the model given by (10).

Consider first $h = 0$. Then

$$Z_t = a_{t-1} + A_{t-1} Z_{t-1} + \xi_t,$$

where

$$\xi_t = \begin{pmatrix} \eta_t \\ 0 \end{pmatrix} = \begin{pmatrix} C_{t-1}^{-1} e_{1,n} \varepsilon_{1t} + C_{t-1}^{-1} I_{2:n} \varepsilon_{2t} \\ 0 \end{pmatrix} = (e_{1,p} \otimes C_{t-1}^{-1} e_{1,n}) \varepsilon_{1t} + e_{1,p} \otimes C_{t-1}^{-1} I_{2:n} \varepsilon_{2t},$$

given that $\eta_t = C_{t-1}^{-1} \varepsilon_t$ and $\varepsilon_t = C_{t-1}^{-1} e_{1,n} \varepsilon_{1t} + C_{t-1}^{-1} I_{2:n} \varepsilon_{2t}$, where $e_{1,n}$ and $I_{2:n}$ are as defined in Section B.2. Hence,

$$y_t = \mathbb{S}_k Z_t = \mathbb{S}_k (e_{1,p} \otimes C_{t-1}^{-1} e_{1,n}) \varepsilon_{1t} + \mathbb{S}_k (a_{t-1} + A_{t-1} Z_{t-1}) + \mathbb{S}_k (e_{1,p} \otimes C_{t-1}^{-1} I_{2:n} \varepsilon_{2t}). \tag{14}$$

Using this decomposition of y_t, we can write $\mathcal{N}_0 = E\left(y_t\varepsilon_{1t}|H_{t-1}=1\right) = \mathcal{N}_{0,1} + \mathcal{N}_{0,2} + \mathcal{N}_{0,3}$, where

$$\begin{aligned}
\mathcal{N}_{0,1} &= E[\mathbb{S}_k(e_{1,p} \otimes C_{t-1}^{-1}e_{1,n})\varepsilon_{1t}^2|H_{t-1}=1], \\
\mathcal{N}_{0,2} &= E[\mathbb{S}_k(a_{t-1} + A_{t-1}Z_{t-1})\varepsilon_{1t}|H_{t-1}=1], \text{ and} \\
\mathcal{N}_{0,3} &= E[\mathbb{S}_k(e_{1,p} \otimes C_{t-1}^{-1}I_{2:n}\varepsilon_{2t})\varepsilon_{1t}|H_{t-1}=1].
\end{aligned}$$

Under Assumption 1 and applying repeatedly the law of iterated expectations (LIE), it can be shown that $\mathcal{N}_{0,2} = \mathcal{N}_{0,3} = 0$, implying that $\mathcal{N}_0 \equiv E\left(y_t\varepsilon_{1t}|H_{t-1}=1\right) = \mathcal{N}_{0,1}$. Thus,

$$\mathcal{N}_0 = \mathbb{S}_k(e_{1,p} \otimes C_E^{-1}e_{1,n})E\left(\varepsilon_{1t}^2|H_{t-1}=1\right).$$

Since $b_h(1) \equiv \mathcal{N}_0\mathcal{D}$, for $h=0$, where $\mathcal{D} \equiv [E\left(\varepsilon_{1t}^2|H_{t-1}=1\right)]^{-1}$, this implies the result. A similar argument shows that

$$\hat{b}_h(0) \xrightarrow{p} b_h(0) = \mathbb{S}_k(e_{1,p} \otimes C_R^{-1}e_{1,n}) \text{ for } h=0.$$

Next, we consider $h=1$. Now,

$$\hat{b}_h(1) \xrightarrow{p} b_h(1) \equiv E\left(y_{t+1}\varepsilon_{1t}|H_{t-1}=1\right)[E\left(\varepsilon_{1t}^2|H_{t-1}=1\right)]^{-1} \equiv \mathcal{N}_1\mathcal{D} \text{ when } h=1.$$

To obtain $\mathcal{N}_1$, we can use the fact that

$$\begin{aligned}
y_{t+1} &= \mathbb{S}_kZ_{t+1} = \mathbb{S}_k(a_t + A_tZ_t + \xi_{t+1}) \\
&= \mathbb{S}_k(a_t + A_t(a_{t-1} + A_{t-1}Z_{t-1} + \xi_t) + \xi_{t+1}) \\
&= \mathbb{S}_kA_t\xi_t + \mathbb{S}_k(a_t + A_t(a_{t-1} + A_{t-1}Z_{t-1})) + \mathbb{S}_k\xi_{t+1}, \qquad (15)
\end{aligned}$$

where $\xi_s = (e_{1,p} \otimes C_{s-1}^{-1}e_{1,n})\varepsilon_{1s} + e_{1,p} \otimes C_{s-1}^{-1}I_{2:n}\varepsilon_{2s}$ for $s = t, t+1$. This implies that $\mathcal{N}_1 \equiv E\left(y_{t+1}\varepsilon_{1t}|H_{t-1}=1\right) = \mathcal{N}_{1,1} + \mathcal{N}_{1,2} + \mathcal{N}_{1,3}$, where

$$\begin{aligned}
\mathcal{N}_{1,1} &= E(\mathbb{S}_kA_t\xi_t\varepsilon_{1t}|H_{t-1}=1), \\
\mathcal{N}_{1,2} &= E[\mathbb{S}_k(a_t + A_t(a_{t-1} + A_{t-1}Z_{t-1}))\varepsilon_{1t}|H_{t-1}=1], \text{ and} \\
\mathcal{N}_{1,3} &= E[\mathbb{S}_k\xi_{t+1}\varepsilon_{1t}|H_{t-1}=1].
\end{aligned}$$

Given the definition of ξ_{t+1}, we can easily see that $\mathcal{N}_{1,3} = 0$ by Assumption B.1, since it implies that $E\left(\xi_{t+1}|\mathcal{F}^t\right) = 0$. To conclude that $\mathcal{N}_{1,2} = 0$, we use the exogeneity condition on H_t, i.e. the fact that $H_t = \eta\left(q_s : s \le t\right)$ with q_s satisfying Assumption B.2. Under these assumptions, H_t and ε_{1t} are mutually independent, implying that by the LIE, we can write

$$\mathcal{N}_{1,2} = E[\mathbb{S}_k(a_t + A_t(a_{t-1} + A_{t-1}Z_{t-1}))E\left(\varepsilon_{1t}|\mathcal{F}^{t-1}, H_t\right)|H_{t-1}=1],$$

where $\mathcal{F}^{t-1} = \sigma(z_{t-1}, H_{t-1}, z_{t-2}, H_{t-2}, \ldots)$. Since $E\left(\varepsilon_{1t}|\mathcal{F}^{t-1}, H_t\right) = E(\varepsilon_{1t}) = 0$, we obtain that $\mathcal{N}_{1,2} = 0$. Hence, $\mathcal{N}_1 = \mathcal{N}_{1,1}$. The result follows because we can show that

$$\mathcal{N}_{1,1} = E[\mathbb{S}_k A_t(e_{1,p} \otimes C_{t-1}^{-1} e_{1,n})\varepsilon_{1t}^2 | H_{t-1} = 1],$$

under Assumption B.1 and B.2. More specifically, using the definition of ξ_t, $\mathcal{N}_{1,1}$ can be decomposed as follows:

$$\mathcal{N}_{1,1} = E[\mathbb{S}_k A_t(e_{1,p} \otimes C_{t-1}^{-1} e_{1,n})\varepsilon_{1t}^2 | H_{t-1} = 1] + E[\mathbb{S}_k A_t(e_{1,p} \otimes C_{t-1}^{-1} I_{2:n}\varepsilon_{2t}\varepsilon_{1t}) | H_{t-1} = 1],$$

where $E\left(\varepsilon_{1t}\varepsilon_{2t}|H_t, \mathcal{F}^{t-1}\right) = E(\varepsilon_{1t}\varepsilon_{2t}) = 0$ under our assumptions. This implies that

$$b_h(1) = \frac{E[\mathbb{S}_k A_t(e_{1,p} \otimes C_{t-1}^{-1} e_{1,n})\varepsilon_{1t}^2 | H_{t-1} = 1]}{E\left(\varepsilon_{1t}^2 | H_{t-1} = 1\right)}.$$

The result follows because the numerator simplifies to $E[\mathbb{S}_k A_t(e_{1,p} \otimes C_{t-1}^{-1} e_{1,n}) | H_{t-1} = 1][E\left(\varepsilon_{1t}^2 | H_{t-1} = 1\right)]$ under the assumption that ε_{1t} is i.i.d.$\left(0, \sigma_1^2\right)$. A similar result holds for $b_h(0)$ when $h = 1$. The proof for other values of h follows from similar arguments.

C Parameters for the data generating process in Section 5

The data generating process in Section 5 uses the following parameter values obtained by fitting the model to the quarterly data used in Ramey and Zubairy (2018), assuming that a recession corresponds to periods when unemployment is above the historical mean:

$$C_E = \begin{bmatrix} 1 & 0 & 0 \\ -0.0097 & 1 & 0 \\ 0.0056 & 0.0371 & 1 \end{bmatrix}, C_R = \begin{bmatrix} 1 & 0 & 0 \\ -0.0495 & 1 & 0 \\ -0.0510 & -0.2134 & 1 \end{bmatrix}, k_E = \begin{bmatrix} 0 \\ 0.0034 \\ 0.0177 \end{bmatrix}, k_R = \begin{bmatrix} 0 \\ 0.0145 \\ 0.1007 \end{bmatrix},$$

$$A_{E,1} = C_E^{-1} B_{E,1} = \begin{bmatrix} -0.1741 & 0 & 0 \\ 0.0317 & 0.8185 & -0.0437 \\ -0.0586 & 0.7540 & 1.4140 \end{bmatrix}, A_{E,2} = \begin{bmatrix} 0.4266 & 0 & 0 \\ 0.1107 & -0.0105 & 0.1177 \\ 0.0296 & -0.7467 & -0.4706 \end{bmatrix},$$

$$A_{E,3} = \begin{bmatrix} 0.4065 & 0 & 0 \\ 0.0889 & 0.2965 & -0.1358 \\ 0.0168 & -0.3586 & 0.0918 \end{bmatrix}, A_{E,4} = \begin{bmatrix} 0.3633 & 0 & 0 \\ 0.0774 & -0.1165 & 0.0595 \\ 0.0535 & 0.3428 & -0.0505 \end{bmatrix},$$

$$A_{R,1} = \begin{bmatrix} 0.2952 & 0 & 0 \\ 0.0088 & 1.6449 & 0.1237 \\ 0.0098 & 0.0450 & 1.4823 \end{bmatrix}, A_{R,2} = \begin{bmatrix} -0.0854 & 0 & 0 \\ 0.0463 & -0.8551 & -0.1995 \\ -0.0051 & -0.0752 & -0.7047 \end{bmatrix},$$

$$A_{R,3} = \begin{bmatrix} 0.1670 & 0 & 0 \\ 0.0107 & 0.2722 & 0.0245 \\ -0.0154 & 0.0911 & 0.2347 \end{bmatrix}, A_{R,4} = \begin{bmatrix} -0.0331 & 0 & 0 \\ -0.0019 & -0.0869 & 0.0410 \\ 0.0476 & -0.0333 & -0.1174 \end{bmatrix}.$$

D Additional simulation results

This appendix contains additional simulation results. Figures D.1 and D.2 report simulation results when $\gamma_E = 0.9$, $\gamma_R = -0.1$ in DGP 1 and DGP 2. Figures D.3 and D.4 report the cumulative government spending multiplier for $\delta \in \{-1, -5, -10\}$.

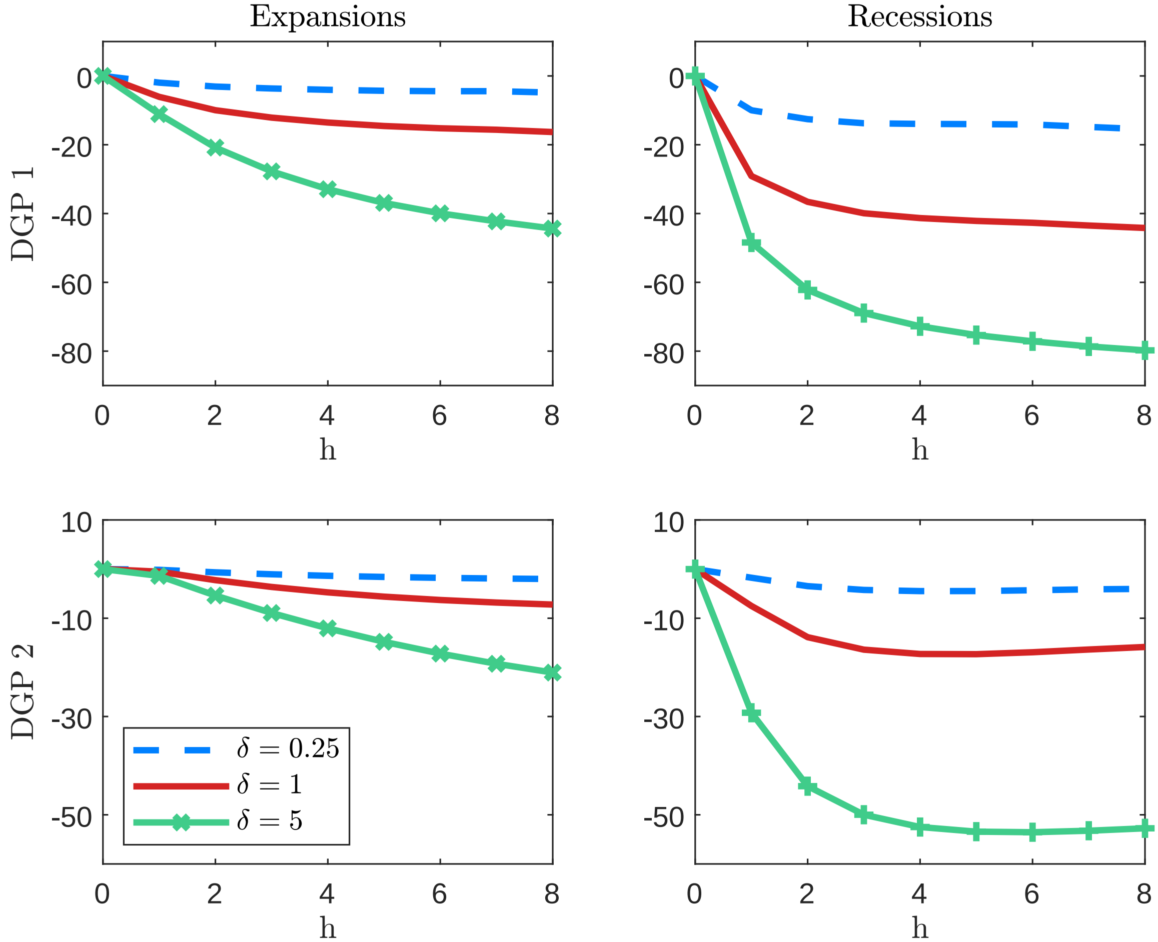

Figure D.1: Asymptotic bias of LP response when $H_t = 1\,(y_t > 0)$

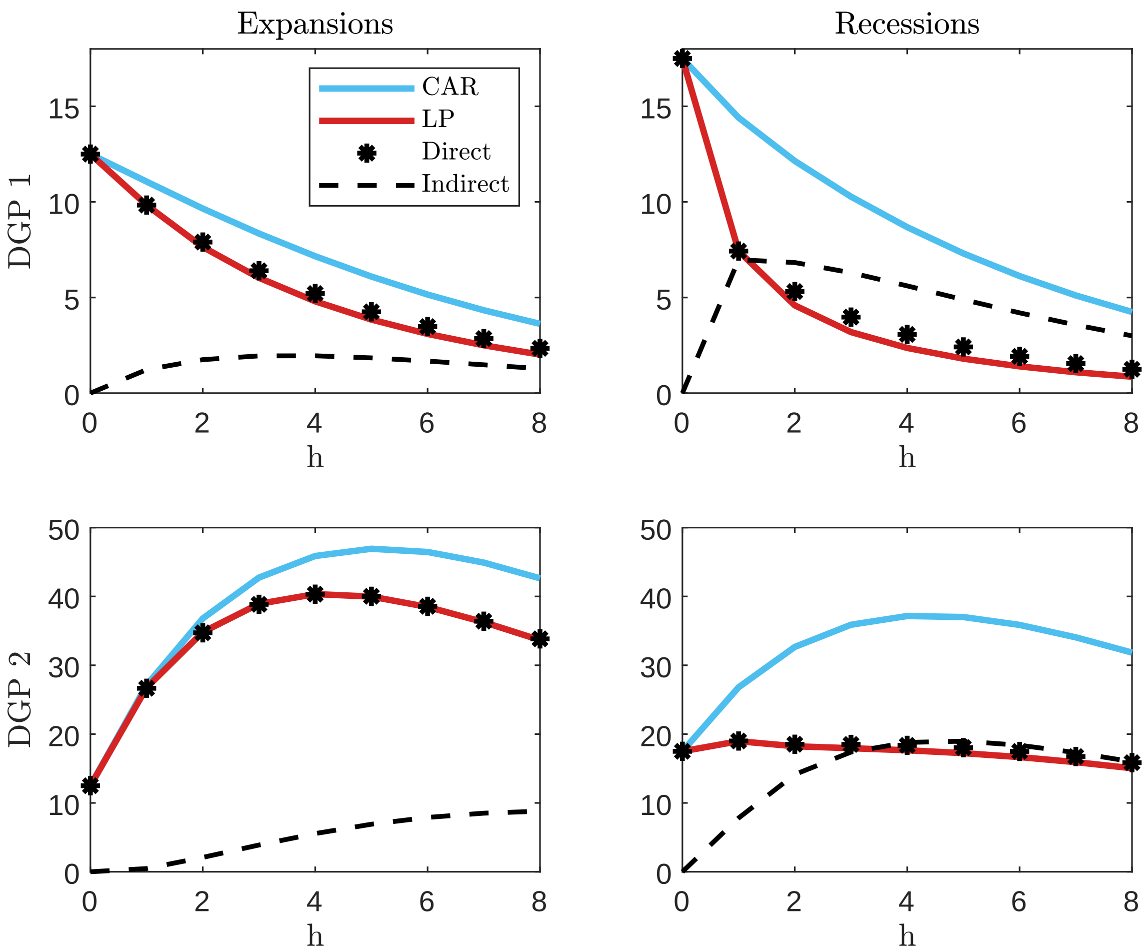

Figure D.2: LP response and decomposition of CAR when $H_t = 1\,(y_t > 0)$ and $\delta = 5$

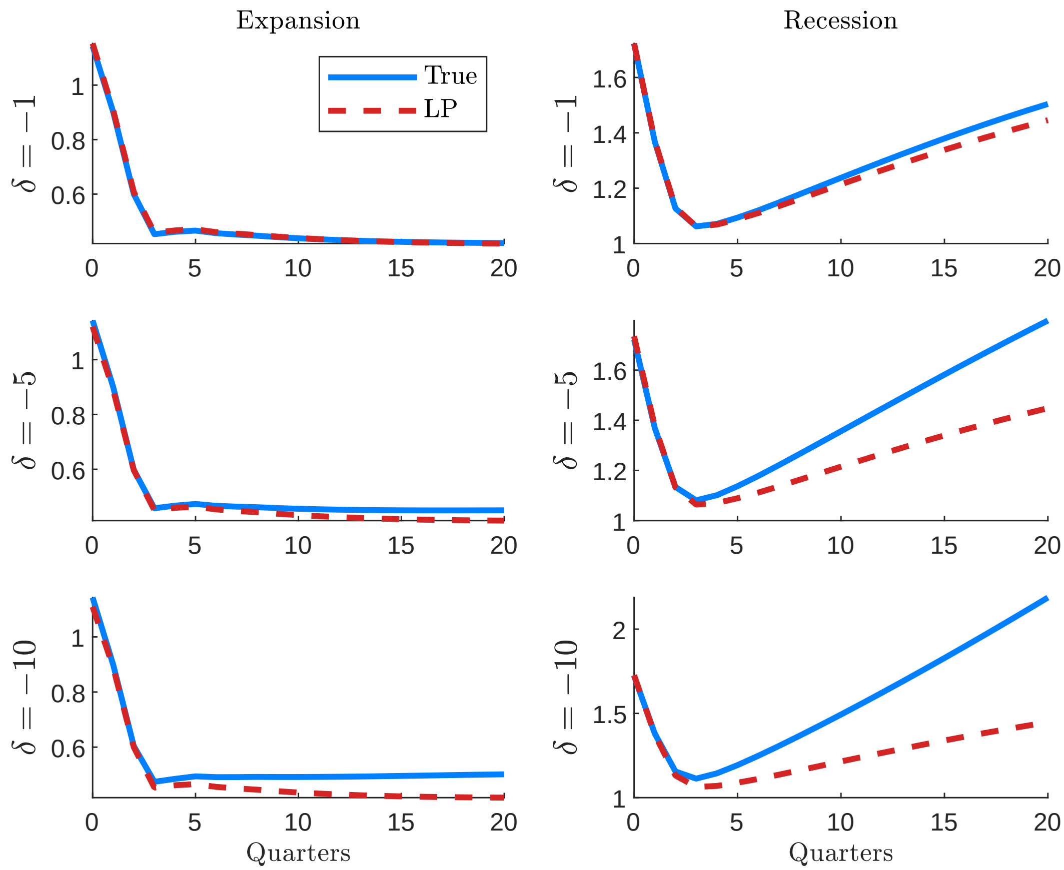

Figure D.3: Cumulative spending multiplier when $H_t = 1\,(y_t > 1)$

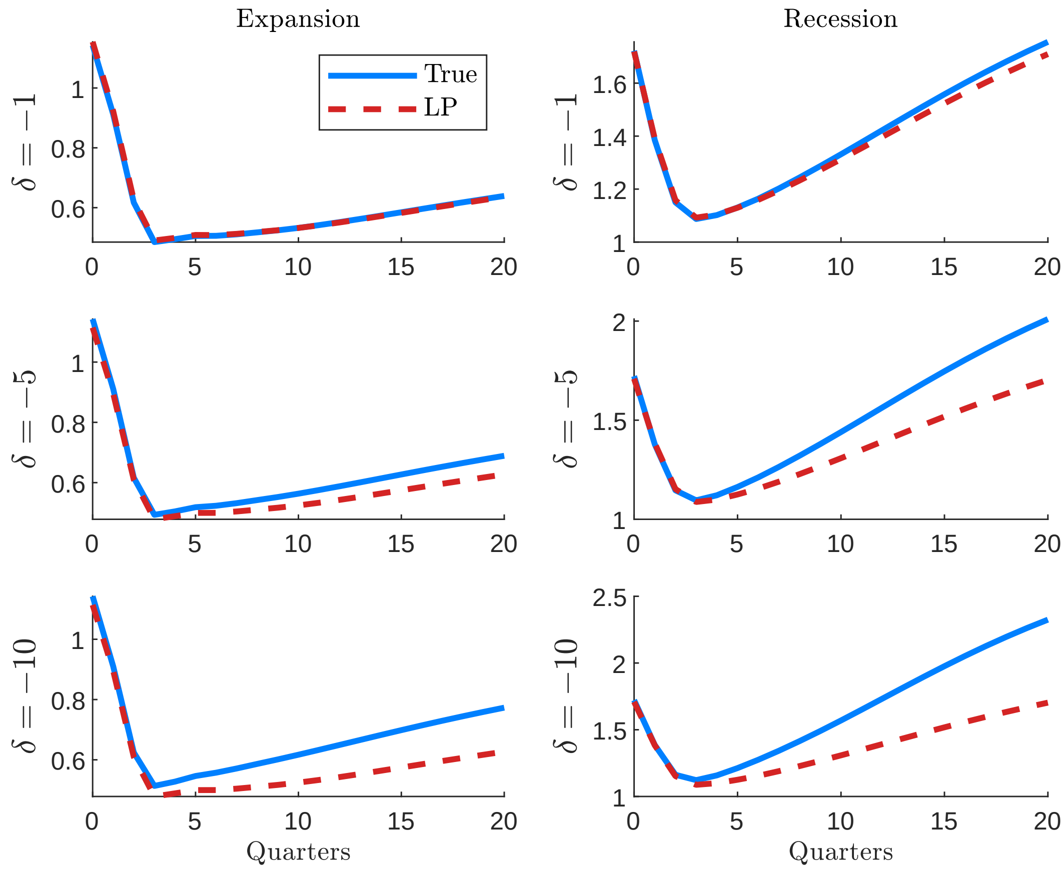

Figure D.4: Cumulative spending multiplier when $H_t = 1\,(y_t > MA(12))$

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